

On trace cohomology

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Abstract

The trace cohomology of an algebraic variety is an abelian subgroup of the real line, obtained from the canonical trace on a C^* -algebra attached to the variety. It is shown that all but one of the Weil conjectures follow from the standard properties of the trace cohomology; the remaining analog of the Riemann hypothesis is proved for reductions of the Kuga-Sato varieties. An explicit formula linking the ℓ -adic and trace cohomologies is introduced.

Key words and phrases: motives, Serre C^* -algebra

MSC: 14F42 (motives); 46L85 (noncommutative topology)

1 Introduction

Let \mathbb{F}_q be a finite field with $q = p^r$ elements and $V := V(\mathbb{F}_q)$ be a smooth n -dimensional projective variety over \mathbb{F}_q . Recall that the Weil conjectures establish a deep connection between the arithmetic of variety V and topology of a variety $V_{\mathbb{C}}$ over the field of complex numbers, defined by the polynomial equations corresponding to V , see [Weil 1949] [18]. To remind the conjectures, let N_m be the number of rational points of V over the field \mathbb{F}_{q^m} and

$$Z_V(t) = \exp \left(\sum_{m=1}^{\infty} N_m \frac{t^m}{m} \right) \quad (1)$$

the corresponding zeta function. Weil conjectured that: (i) $Z_V(t)$ is a quotient of polynomials with rational coefficients; (ii) $Z_V(q^{-n}t^{-1}) = \pm q^{n\frac{x}{2}} t^x Z_V(t)$,

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where χ is the Euler-Poincaré characteristic of $V_{\mathbb{C}}$; (iii) $Z_V(t)$ satisfies an analog of the Riemann hypothesis, i.e.

$$Z_V(t) = \frac{P_1(t)P_3(t) \dots P_{2n-1}(t)}{P_0(t)P_2(t) \dots P_{2n}(t)}, \quad (2)$$

so that $P_0(t) = 1 - t$, $P_{2n}(t) = 1 - q^n t$ and for each $1 \leq i \leq 2n - 1$ the polynomial $P_i(t)$ has integer coefficients and can be written in the form $P_i(t) = \prod(1 - \alpha_{ij}t)$, where α_{ij} are algebraic integers with $|\alpha_{ij}| = q^{\frac{i}{2}}$; (iv) the degree of polynomial $P_i(t)$ is equal to the i -th Betti number of variety $V_{\mathbb{C}}$. Weil verified his conjectures for the algebraic curves (i.e. for $n = 1$) and pointed out that in general the Weil conjectures follow from an appropriate cohomology theory for variety V . (Such a remark is based on the fact that rational points of V are fixed points of the Frobenius endomorphism of V ; the latter induces a linear map on the cohomology so that the number N_m can be easily found using linear algebra.) Such a cohomology was constructed by Grothendieck and called the ℓ -adic cohomology; all but conjecture (iii) can be deduced from the basic properties of the ℓ -adic cohomology and item (iii) depends on the yet unproved Standard Conjectures on algebraic cycles, see [Grothendieck 1968] [6]. Since each prime $\ell \neq \text{char}(\mathbb{F}_q)$ gives rise to a cohomology theory of V , it was conjectured by Grothendieck that there exists a universal cohomology called a *motive* of V , such that the ℓ -adic cohomology is a realization of the motive for any ℓ , see *ibid.*, p. 198.

The aim of our note is a cohomology theory implying all Weil conjectures for the (reduction of) Kuga-Sato varieties and universal with respect to the ℓ -adic cohomology. Recall that the Serre C^* -algebra \mathcal{A}_V of projective variety $V_{\mathbb{C}}$ is the norm-closure of a self-adjoint representation of the twisted homogeneous coordinate ring of $V_{\mathbb{C}}$ by the bounded linear operators on a Hilbert space \mathcal{H} , see Section 2 for an exact definition. We shall write $\tau : \mathcal{A}_V \otimes \mathcal{K} \rightarrow \mathbb{R}$ to denote the canonical normalized trace on the stable C^* -algebra $\mathcal{A}_V \otimes \mathcal{K}$, i.e. a positive linear functional of norm 1 such that $\tau(yx) = \tau(xy)$ for all $x, y \in \mathcal{A}_V \otimes \mathcal{K}$, see [Blackadar 1986] [2], p. 31. Applying the Chern character formula to the algebra $\mathcal{A}_V \otimes \mathcal{K}$, one obtains for each $0 \leq i \leq 2n$ an induced by τ injective homomorphism

$$\tau_* : H^i(V_{\mathbb{C}}) \longrightarrow \mathbb{R}, \quad (3)$$

see Section 2 for the details. By a *trace cohomology* $H_{tr}^i(V)$ of variety V one understands the additive abelian subgroup $\tau_*(H^i(V_{\mathbb{C}}))$ of the real line \mathbb{R} . Our main results can be formulated as follows.

Theorem 1 *The Weil conjectures (i)-(iv) for the Kuga-Sato varieties follow from the standard properties of the trace cohomology.*

Theorem 2 *The ℓ -adic cohomology $H_{et}^i(V; \mathbf{Q}_\ell)$ is a pull back of the tensor product $H_{tr}^i(V) \otimes \mathbf{Q}_\ell$ for each prime $\ell \neq \text{char}(\mathbb{F}_q)$.*

Remark 1 Theorem 2 says that the trace cohomology is universal, i.e. each ℓ -adic cohomology group can be obtained from $H_{tr}^i(V)$ by closing arrows of the diagram in Fig. 1, where ι is the natural inclusion map.

$$\begin{array}{ccc}
 H^i(V_{\mathbb{C}}) & \xhookrightarrow{\iota} & H_{et}^i(V; \mathbf{Q}_\ell) \\
 \searrow \tau_* & & \nearrow \\
 & H_{tr}^i(V) \otimes \mathbf{Q}_\ell &
 \end{array}$$

Figure 1: Universal property of trace cohomology.

The article is organized as follows. The Serre C^* -algebras and trace cohomology are introduced in Section 2. Theorems 1 and 2 are proved in Section 3. The trace cohomology of the algebraic (and elliptic, in particular) curves is calculated in Section 4.

2 Preliminaries

The basics of non-commutative algebraic geometry can be found in a survey by [Stafford & van den Bergh 2001] [17]. For an introduction to the C^* -algebras and their K -theory we refer the reader to [Murphy 1990] [10] and [Blackadar 1986] [2], respectively. The Serre C^* -algebras were defined in [12] and the trace cohomology in [13].

2.1 Serre C^* -algebra

Let V be a projective scheme over a field k and let \mathcal{L} be the invertible sheaf of linear forms on V . If σ is an automorphism of V , then the pullback of

\mathcal{L} along σ will be denoted by \mathcal{L}^σ , i.e. $\mathcal{L}^\sigma(U) := \mathcal{L}(\sigma U)$ for every $U \subset V$. Consider the graded k -algebra

$$B(V, \mathcal{L}, \sigma) = \bigoplus_{i \geq 0} H^0(V, \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \dots \otimes \mathcal{L}^{\sigma^{i-1}}) \quad (4)$$

called a *twisted homogeneous coordinate ring* of V ; notice that such a ring is non-commutative unless σ is the trivial automorphism. Recall that multiplication of sections of $B(V, \mathcal{L}, \sigma)$ is defined by the rule $ab = a \otimes b^{\sigma^m}$, where $a \in B_m$ and $b \in B_n$. Given a pair (V, σ) consisting of a Noetherian scheme V and an automorphism σ of V , an invertible sheaf \mathcal{L} on V is called σ -*ample*, if for every coherent sheaf \mathcal{F} on V , the cohomology group $H^q(V, \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \dots \otimes \mathcal{L}^{\sigma^{n-1}} \otimes \mathcal{F})$ vanishes for $q > 0$ and $n \gg 0$. Notice, that if σ is trivial, this definition is equivalent to the usual definition of ample invertible sheaf [Serre 1955] [15]. If \mathcal{L} is a σ -ample invertible sheaf on V , then

$$\mathbf{Mod} (B(V, \mathcal{L}, \sigma)) / \mathbf{Tors} \cong \mathbf{Coh} (V), \quad (5)$$

where \mathbf{Mod} is the category of graded left modules over the ring $B(V, \mathcal{L}, \sigma)$, \mathbf{Tors} is the full subcategory of \mathbf{Mod} of the torsion modules and \mathbf{Coh} is the category of quasi-coherent sheaves on a scheme V , see [M. Artin & van den Bergh 1990] [1]. In view of (5) the ring $B(V, \mathcal{L}, \sigma)$ is indeed a coordinate ring of V , see [Serre 1955] [15].

Remark 2 Suppose that R is a commutative graded ring, such that $V = \text{Spec} (R)$ and denote by $R[t, t^{-1}; \sigma]$ the ring of skew Laurent polynomials defined by the commutation relation $b^\sigma t = tb$ for all $b \in R$, where b^σ is the image of b under automorphism $\sigma : V \rightarrow V$; then $R[t, t^{-1}; \sigma] \cong B(V, \mathcal{L}, \sigma)$, see [M. Artin & van den Bergh 1990] [1].

Let \mathcal{H} be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on \mathcal{H} . For a ring of skew Laurent polynomials $R[t, t^{-1}; \sigma]$, we shall consider a homomorphism

$$\rho : R[t, t^{-1}; \sigma] \longrightarrow \mathcal{B}(\mathcal{H}). \quad (6)$$

Recall that algebra $\mathcal{B}(\mathcal{H})$ is endowed with a $*$ -involution coming from the scalar product on the Hilbert space \mathcal{H} . We shall call representation (6) *$*$ -coherent* if (i) $\rho(t)$ and $\rho(t^{-1})$ are unitary operators, such that $\rho^*(t) = \rho(t^{-1})$ and (ii) for all $b \in R$ it holds $(\rho^*(b))^{\sigma(\rho)} = \rho^*(b^\sigma)$, where $\sigma(\rho)$ is an automorphism of $\rho(R)$ induced by σ . Whenever $B := R[t, t^{-1}; \sigma]$ admits a $*$ -coherent representation, $\rho(B)$ is a $*$ -algebra; the norm-closure of $\rho(B)$ yields a C^* -algebra, see e.g. [Murphy 1990] [10], Section 2.1.

Definition 1 By a Serre C^* -algebra of V we understand the norm-closure of $\rho(B)$; such a C^* -algebra will be denoted by \mathcal{A}_V .

Remark 3 Each Serre C^* -algebra \mathcal{A}_V is a crossed product C^* -algebra, see e.g. [Williams 2007] [19], pp 47-54 for the definition and details; namely, $\mathcal{A}_V \cong C(V) \rtimes_{\sigma} \mathbb{Z}$, where $C(V)$ is the C^* -algebra of all continuous complex-valued functions on V and σ is a $*$ -coherent automorphism of V .

Remark 4 Let \mathcal{K} be the C^* -algebra of all compact operators on a Hilbert space \mathcal{H} . The stable Serre C^* -algebra $\mathcal{A}_V \otimes \mathcal{K}$ is endowed with the unique normalized trace (tracial state) $\tau : \mathcal{A}_V \otimes \mathcal{K} \rightarrow \mathbb{R}$, i.e. a positive linear functional of norm 1 such that $\tau(yx) = \tau(xy)$ for all $x, y \in \mathcal{A}_V \otimes \mathcal{K}$, see [Blackadar 1986] [2], p. 31.

2.2 Trace cohomology

Let k be a number field. Let $V(k)$ be a smooth n -dimensional projective variety over k , such that variety $V := V(\mathbb{F}_q)$ is the reduction modulo q of $V(k)$; in other words, $V(k)$ is defined by polynomial equations for V over the field of complex numbers. Because the Serre C^* -algebra \mathcal{A}_V of $V(k)$ is a crossed product C^* -algebra of the form $\mathcal{A}_V \cong C(V(k)) \rtimes \mathbb{Z}$ (Remark 3), one can use the Pimsner-Voiculescu six term exact sequence for the crossed products, see e.g. [Blackadar 1986] [2], p. 83 for the details. Thus one gets the short exact sequence of the algebraic K -groups: $0 \rightarrow K_0(C(V(k))) \xrightarrow{i_*} K_0(\mathcal{A}_V) \rightarrow K_1(C(V(k))) \rightarrow 0$, where map i_* is induced by the natural embedding of $C(V(k))$ into \mathcal{A}_V . We have $K_0(C(V(k))) \cong K^0(V(k))$ and $K_1(C(V(k))) \cong K^{-1}(V(k))$, where K^0 and K^{-1} are the topological K -groups of variety $V(k)$, see [Blackadar 1986] [2], p. 80. By the Chern character formula, one gets $K^0(V(k)) \otimes \mathbb{Q} \cong H^{even}(V(k); \mathbb{Q})$ and $K^{-1}(V(k)) \otimes \mathbb{Q} \cong H^{odd}(V(k); \mathbb{Q})$, where H^{even} (H^{odd}) is the direct sum of even (odd, resp.) cohomology groups of $V(k)$. Notice that $K_0(\mathcal{A}_V \otimes \mathcal{K}) \cong K_0(\mathcal{A}_V)$ because of stability of the K_0 -group with respect to tensor products by the algebra \mathcal{K} , see e.g. [Blackadar 1986] [2], p. 32. One gets the commutative diagram in Fig. 2, where τ_* denotes a homomorphism induced on K_0 by the canonical trace τ on the C^* -algebra $\mathcal{A}_V \otimes \mathcal{K}$. Because $H^{even}(V(k)) := \bigoplus_{i=0}^n H^{2i}(V(k))$ and $H^{odd}(V) := \bigoplus_{i=1}^n H^{2i-1}(V(k))$, one gets for each $0 \leq i \leq 2n$ an injective homomorphism

$$\tau_* : H^i(V(k)) \longrightarrow \mathbb{R}. \quad (7)$$

$$\begin{array}{ccc}
H^{even}(V(k)) \otimes \mathbb{Q} & \xrightarrow{i_*} & K_0(\mathcal{A}_V \otimes \mathcal{K}) \otimes \mathbb{Q} \longrightarrow H^{odd}(V(k)) \otimes \mathbb{Q} \\
& \searrow & \downarrow \tau_* \swarrow \\
& & \mathbb{R}
\end{array}$$

Figure 2: The trace cohomology.

Definition 2 By an i -th trace cohomology group $H_{tr}^i(V)$ of V one understands the abelian subgroup of \mathbb{R} defined by map (7).

Remark 5 The abelian group $H_{tr}^i(V)$ is called a *pseudo-lattice*, see [Manin 2004] [8], Section 1. The endomorphisms in the category of pseudo-lattices are given by multiplication of its points by the real numbers α such that $\alpha H_{tr}^i(V) \subseteq H_{tr}^i(V)$. It is known that the ring $End(H_{tr}^i(V)) \cong \mathbb{Z}$ or $End(H_{tr}^i(V)) \otimes \mathbb{Q}$ is a real algebraic number field. In the latter case $H_{tr}^i(V) \subset End(H_{tr}^i(V)) \otimes \mathbb{Q}$, see [Manin 2004] [8], Lemma 1.1.1 for the case of quadratic fields. Notice that one can write multiplication by α in a matrix form by fixing a basis in the pseudo-lattice; thus the ring $End(H_{tr}^i(V))$ is a commutative subring of the matrix ring $M_{b_i}(\mathbb{Z})$, where b_i is equal to the rank of pseudo-lattice, i.e. the cardinality of its basis.

Remark 6 Notice that the trace cohomology $H_{tr}^i(V)$ is a totally ordered abelian group, see e.g. [Goodearl 1986] [5] for an introduction to this area; the order is the natural order on the set \mathbb{R} .

3 Proofs

3.1 Proof of theorem 1

Conjectures (i), (ii) and (iv) were proved in [13]; here we shall focus on Weil's conjecture (iii), i.e. an analog of the Riemann hypothesis for the function $Z_V(t)$. For the sake of clarity, let us outline main ideas. The trace cohomology $H_{tr}^i(V)$ will be used to construct a positive-definite Hermitian form $\varphi(x, y)$ on

the cohomology group $H^i(V(k))$. Such a construction involves the Deligne-Scholl theory linking the ℓ -adic cohomology $H_{et}^i(V; \mathbf{Q}_\ell)$ of the Kuga-Sato variety V with the space of cusp forms $S_{i+1}(\Gamma)$ of weight $i + 1$ for a finite index subgroup $\Gamma \subset SL_2(\mathbb{Z})$, see [Deligne 1969] [3] and [Scholl 1985] [14]. It is proved that the Petersson inner product on S_{i+1} defines, via the trace cohomology, the required form $\varphi(x, y)$. Since the regular maps of V preserve the form $\varphi(x, y)$ modulo a positive constant, one obtains an analog of the Riemann hypothesis for the zeta function of V . We shall split the proof in a series of lemmas.

Lemma 1 (Deligne-Scholl) *If V is the Kuga-Sato variety, then there exists a finite index subgroup Γ of the modular group $SL_2(\mathbb{Z})$, such that*

$$\dim H_{tr}^i(V) = 2 \dim_{\mathbb{C}} S_{i+1}(\Gamma), \quad (8)$$

where $S_{i+1}(\Gamma)$ is the space of cusp forms of weight $i + 1$ relatively group Γ .

Proof. This lemma follows from the results of [Deligne 1969] [3] and [Scholl 1985] [14]. Namely, let Γ be a finite index subgroup of $SL_2(\mathbb{Z})$, such that the modular curve $X_\Gamma := \mathbb{H}/\Gamma$ can be defined over the field \mathbb{Q} . It was proved that for each prime ℓ there exists a continuous homomorphism

$$\rho : Gal(\bar{\mathbb{Q}} | \mathbb{Q}) \rightarrow End(W), \quad (9)$$

where W is a $2d$ -dimensional vector space over ℓ -adic numbers \mathbf{Q}_ℓ and $d = \dim_{\mathbb{C}} S_{i+1}(\Gamma)$, see [Scholl 1985] [14]. It was proved earlier, that $W \cong H_{et}^i(V; \mathbf{Q}_\ell)$ for a variety V over the field \mathbb{Q} and the (arithmetic) Frobenius element of the Galois group $Gal(\bar{\mathbb{Q}} | \mathbb{Q})$ corresponds to the (geometric) Frobenius endomorphism of the ℓ -adic cohomology $H_{et}^i(V; \mathbf{Q}_\ell)$, see [Deligne 1969] [3] for Γ being a congruence group.

Let $V(k)$ be a variety over the complex numbers associated to V . By the comparison theorem

$$H_{et}^i(V; \mathbf{Q}_\ell) \otimes_{\mathbf{Q}_\ell} \mathbb{C} \cong H^i(V(k); \mathbb{C}), \quad (10)$$

see e.g. [Hartshorne 1977] [7], p. 454. On the other hand, from definition 2 we have $\dim H_{tr}^i(V) = \dim H^i(V(k); \mathbb{C})$ and, therefore,

$$\dim H_{tr}^i(V) = \dim H_{et}^i(V). \quad (11)$$

But $\dim H_{et}^i(V; \mathbf{Q}_\ell) = 2 \dim_{\mathbb{C}} S_{i+1}(\Gamma)$ by the Deligne-Scholl theory and, therefore,

$$\dim H_{tr}^i(V) = 2 \dim_{\mathbb{C}} S_{i+1}(\Gamma). \quad (12)$$

Lemma 1 follows. \square

Remark 7 Strictly speaking, the above lemma is true for the Kuga-Sato varieties only; however, the author wishes to express a cautious hope that similar result is true for a wider class of varieties.

Lemma 2 *The trace cohomology $H_{tr}^i(V)$ defines a canonical \mathbb{Z} -module embedding*

$$H^i(V(k)) \hookrightarrow S_{i+1}(\Gamma). \quad (13)$$

Proof. Recall that the *Petersson inner product*

$$(x, y) : S_{i+1}(\Gamma) \times S_{i+1}(\Gamma) \rightarrow \mathbb{C} \quad (14)$$

on the space $S_{i+1}(\Gamma)$ is given by the integral

$$(f, g) = \int_{X_\Gamma} f(z) \overline{g(z)} (\Im z)^{i+1} dz. \quad (15)$$

The product is linear in f and conjugate-linear in g , so that $(g, f) = \overline{(f, g)}$ and $(f, f) > 0$ for all $f \neq 0$, see e.g. [Milne 1997] [9], p. 57.

Fix a basis $\{\alpha_1, \dots, \alpha_d; \beta_1, \dots, \beta_d\}$ in the \mathbb{Z} -module $H_{tr}^i(V)$. In view of the standard properties of scalar product (x, y) , there exists a unique cusp form $g \in S_{i+1}(\Gamma)$, such that

$$(f_j, g) = \alpha_j + i\beta_j, \quad (16)$$

where $\{f_1, \dots, f_d\}$ is the orthonormal basis in $S_{i+1}(\Gamma)$ consisting of the Hecke eigenforms. But α_j and β_j are the image of generators of the \mathbb{Z} -module $H^i(V(k))$ under the trace map τ_* , see definition 2. Thus the pull back of map (16) defines an embedding

$$\iota : H^i(V(k)) \hookrightarrow S_{i+1}(\Gamma), \quad (17)$$

whose image $\iota(H^i(V(k)))$ is a \mathbb{Z} -module generated by the real and imaginary parts of the Hecke eigenforms $f_j \in S_{i+1}(\Gamma)$. Lemma 2 follows. \square

Corollary 1 *There exists a positive-definite Hermitian form*

$$\varphi(x, y) : H^i(V(k)) \times H^i(V(k)) \rightarrow \mathbb{C}, \quad (18)$$

on the \mathbb{Z} -module $H^i(V(k))$ coming from the Petersson inner product on the space $S_{i+1}(\Gamma)$.

Proof. The Petersson inner product is a Hermitian form because $(g, f) = \overline{(f, g)}$ and a positive-definite form because $(f, f) > 0$ for all $f \neq 0$. In view of lemma 2, one gets the conclusion of corollary 1. \square

Remark 8 The ring $\text{End}(H_{tr}^i(V))$ is isomorphic to the ring $\mathbb{T}_{i+1}(\Gamma)$ of the Hecke operators on the space $S_{i+1}(\Gamma)$ and it is a commutative subring of the matrix ring $\text{End}(H^i(V(k)))$.

Proof of remark 8. In view of lemma 1 and remark 5, the ring $\text{End}(H_{tr}^i(V))$ is generated by the eigenvalues of Hecke operators corresponding to their (common) eigenform $f_j \in S_{i+1}(\Gamma)$; notice that $\mathbb{T}_{i+1}(\Gamma)$ is always a non-trivial ring if Γ is a congruence subgroup and extends to such for the non-congruence subgroups of finite index as shown in [Scholl 1985] [14]. On the other hand, it is known that the Hecke ring $\mathbb{T}_{i+1}(\Gamma)$ is isomorphic to a commutative subring of the matrix ring $\text{End}(H^i(V(k)))$ represented by the symmetric matrices with positive integer entries, see e.g. [Milne 1997] [9]. Remark 8 follows. \square

Lemma 3 *Each regular map $f : V \rightarrow V$ induces a linear map $f_*^i : H^i(V(k)) \rightarrow H^i(V(k))$ of degree $\deg(f_*)^i$, whose characteristic polynomial $\text{char}(f_*^i)$ has integer coefficients and roots of the absolute value $|\lambda| = [\deg(f_*)^i]^{\frac{1}{2n}}$.*

Proof. Consider a regular map $f : V \rightarrow V$ obtained by the reduction modulo q of an algebraic map $\tilde{f} : V(k) \rightarrow V(k)$ of the corresponding variety over the field of complex numbers. Let us show that the linear map $f_*^i : H^i(V(k)) \rightarrow H^i(V(k))$ induced by \tilde{f} on the integral cohomology $H^i(V(k))$ must preserve, up to a constant multiple, the positive-definite Hermitian form $\varphi(x, y)$ on $H^i(V(k))$ given by formula (18).

Indeed, because $\tilde{f}(V(k)) \subseteq V(k)$ is an algebraic variety of its own, one can repeat the trace cohomology construction for $\tilde{f}(V(k))$; thus one gets a positive-definite Hermitian form $\tilde{\varphi}(x, y)$ on $H^i(\tilde{f}(V(k)))$. But $H^i(\tilde{f}(V(k))) \subseteq H^i(V(k))$ and therefore one gets yet another positive-definite Hermitian form $\varphi(x, y)$ on $H^i(\tilde{f}(V(k)))$. Since such a form is unique (see lemma 2), one

concludes that $\tilde{\varphi}(x, y)$ coincides with $\varphi(x, y)$ modulo a positive factor C . We leave it to the reader to prove that $C = [\deg (f_*^i)]^{\frac{1}{n}}$. (Indeed, the volume form can be calculated by the formula $v = |\det (f_*^i)|v_0 = \deg (f_*^i)v_0$; on the other hand, the multiplication by C map gives the volume $v = C^n v_0$, where n is the dimension of variety V .)

Let λ be a root of the characteristic polynomial $\text{char} (f_*^i) := \det (\lambda I - f_*^i)$. Since the kernel of the map $\lambda I - f_*^i$ is non-trivial, let $x \in \text{Ker} (\lambda I - f_*^i)$ be a non-zero element; clearly, $f_*^i x = \lambda x$. Consider the value of scalar product $(x, y) = \varphi(x, y)$ on $x = y = f_*^i x$, i.e.

$$(f_*^i x, f_*^i x) = (\lambda x, \lambda x) = \lambda \bar{\lambda}(x, x). \quad (19)$$

On the other hand,

$$(f_*^i x, f_*^i x) = [\deg (f_*^i)]^{\frac{1}{n}}(x, x). \quad (20)$$

Because $(x, x) \neq 0$, one can cancel it in (19) and (20), so that

$$\lambda \bar{\lambda} = [\deg (f_*^i)]^{\frac{1}{n}} \quad \text{or} \quad |\lambda| = [\deg (f_*^i)]^{\frac{1}{2n}}. \quad (21)$$

Note that $\text{char} (f_*^i) \in \mathbb{Z}[\lambda]$ because $H^i(V(k))$ is a \mathbb{Z} -module; lemma 3 follows. \square

Remark 9 Note that any non-trivial map $f_*^i \in \text{End} (H_{tr}^i(V)) \subset \text{End} (H^i(V(k)))$ corresponds to a non-algebraic (transcendental) map $\tilde{f} : V(k) \rightarrow V(k)$, because the roots of $\text{char} (f_*^i)$ are real numbers in this case. Of course, there are many other examples of the non-algebraic maps $\tilde{f} : V(k) \rightarrow V(k)$.

Lemma 4 $\deg (f_*^i) = [\deg (f)]^i$.

Proof. It is well known, that the cusp forms $g(z) \in S_{i+1}(\Gamma)$ are bijective with the holomorphic differentials

$$g(z)dz^{\frac{i+1}{2}} \quad (22)$$

on the Riemann surface $X_\Gamma = \mathbb{H}/\Gamma$. To prove lemma 4, one can use the Riemann-Hurwitz formula:

$$2g(Y) - 2 = m [2g(X) - 2] + \sum_P (e_P - 1), \quad (23)$$

where e_P is the multiplicity at the point P of an m -fold holomorphic map $Y \rightarrow X$ between the Riemann surfaces of genus $g(Y)$ and $g(X)$, see e.g.

[Milne 1997] [9], p. 17. Because the differential (22) is locally defined, one can substitute in (23) $g(X) = g(Y) = 0$ and assume $P = 0$ to be a unique ramification point. Thus

$$m = \frac{e_P + 1}{2} \quad (24)$$

and the m -fold differential (22) implies $e_P = i$, i.e. the holomorphic map $Y \rightarrow X$ is given by the formula

$$z \mapsto z^i. \quad (25)$$

On the other hand, for a regular map $f : V \rightarrow V$ it holds $\deg(f) = \deg(\tilde{f}) = \deg(f_*)$. Since degree is a multiplicative function on composition of maps, one gets from (25) and the link between i -th cohomology of $V(k)$ and the space $S_{i+1}(\Gamma)$, that

$$\deg(f_*^i) = [\deg(f_*)]^i = [\deg(f)]^i. \quad (26)$$

Lemma 4 follows. \square

Corollary 2 (Riemann hypothesis) *The roots α_{ij} of polynomials $P_i(t)$ in formula (2) are algebraic numbers of the absolute value $|\alpha_{ij}| = q^{\frac{i}{2}}$.*

Proof. It is easy to see, that the Frobenius map $f : (z_1, \dots, z_n) \mapsto (z_1^q, \dots, z_n^q)$ of variety V is regular and $\deg(f) = q^n$. Therefore, one can apply lemmas 3 and 4 to such a map and get the equality $|\alpha_{ij}| = q^{\frac{i}{2}}$ for each $0 \leq i \leq 2n - 1$. Corollary 2 follows. \square

Corollary 2 finishes the proof of theorem 1. \square

3.2 Proof of theorem 2

For a prime $\ell \neq \text{char}(\mathbb{F}_q)$ denote by $T_\ell \in \mathbb{T}_{i+1}(\Gamma)$ the Hecke operator on the space $S_{i+1}(\Gamma)$. Recall that in a basis of the orthogonal Hecke eigenforms, the operator T_ℓ is given by a positive integer (symmetric) matrix of determinant ℓ . Consider a *dimension group* G_ℓ on $H^i(V(k)) \cong \mathbb{Z}^{2d}$ generated by matrix T_ℓ , i.e.

$$\mathbb{Z}^{2d} \xrightarrow{T_\ell} \mathbb{Z}^{2d} \xrightarrow{T_\ell} \mathbb{Z}^{2d} \xrightarrow{T_\ell} \dots \quad (27)$$

(We refer the reader to [Effros 1981] [4] and [Goodearl 1986] [5] for the definition, origin and basic properties of such groups.)

Lemma 5 $\tau_*(G_\ell) \cong H_{tr}^i(V) \otimes \mathbf{Q}_\ell$.

Proof. Let $\lambda_\ell > 1$ be the Perron-Frobenius eigenvalue of matrix T_ℓ . It is well known, that dimension group G_ℓ is order-isomorphic to an additive abelian subgroup $\mathbb{Z}[\frac{1}{\lambda_\ell}]$ of the real line \mathbb{R} , see e.g. [Effros 1981] [4], p.37. In other words,

$$\tau_*(G_\ell) \cong \mathbb{Z} \left[\frac{1}{\lambda_\ell} \right]. \quad (28)$$

On the other hand, because $T_\ell \in \mathbb{T}_{i+1}(\Gamma) \cong \text{End}(H_{tr}^i(V))$ (see remark 8), one gets the following isomorphisms:

$$\mathbb{Z} \left[\frac{1}{\lambda_\ell} \right] \cong \mathbb{Z} \left[\frac{1}{\lambda} \right] \otimes \mathbb{Z} \left[\frac{1}{\ell} \right] \cong H_{tr}^i(V) \otimes \mathbb{Z} \left[\frac{1}{\ell} \right], \quad (29)$$

where λ is an algebraic unit, such that $\lambda_\ell = \ell \lambda$. But the p -adic closure of the set $\mathbb{Z}[\frac{1}{\ell}]$ is nothing but the field \mathbf{Q}_ℓ of p -adic numbers. Thus one gets

$$\mathbb{Z} \left[\frac{1}{\lambda_\ell} \right] \cong H_{tr}^i(V) \otimes \mathbf{Q}_\ell. \quad (30)$$

Lemma 5 follows from formulas (28) and (30). \square

Lemma 6 $G_\ell \cong H_{et}^i(V; \mathbf{Q}_\ell)$.

Proof. Recall that by definition $\tau_*(H^i(V(k))) \cong H_{tr}^i(V)$. Using lemma 5 and the pull back along the trace map τ_* , one gets from the above formula that $G_\ell \cong H^i(V(k)) \otimes \mathbf{Q}_\ell$. But

$$H^i(V(k)) \otimes \mathbf{Q}_\ell \cong \lim_{m \rightarrow \infty} H^i(V(k); \mathbb{Z}/\ell^m \mathbb{Z}) := H_{et}^i(V; \mathbf{Q}_\ell). \quad (31)$$

Thus $G_\ell \cong H_{et}^i(V; \mathbf{Q}_\ell)$ as an additive abelian group. Lemma 6 follows. \square

Theorem 2 follows from lemmas 5 and 6. \square

Remark 10 The reader familiar with the notion of a shift automorphism (see e.g. [Effros 1981] [4], pp. 37-38) can verify that for each $\ell \neq \text{char}(\mathbb{F}_q)$ the shift automorphism of group $\tau_*(G_\ell)$ corresponds to the action of Frobenius element of the group $\text{Gal}(\bar{k} | k)$ on ℓ -adic cohomology $H_{et}^i(V; \mathbf{Q}_\ell)$. Note that whenever $\ell = \text{char}(\mathbb{F}_q)$ the shift automorphism is trivial and, therefore, one gets an analog of the crystalline cohomology in this case.

4 Examples

The groups $H_{tr}^i(V)$ are truly concrete and simple; in this section we calculate the trace cohomology for $n = 1$, i.e. when V is a smooth algebraic curve. In particular, we find the cardinality of the set $\mathcal{E}(\mathbb{F}_q)$ obtained by the reduction modulo q of an elliptic curve with complex multiplication.

Example 1 The trace cohomology of smooth algebraic curve $\mathcal{C}(\mathbb{F}_q)$ of genus $g \geq 1$ is given by the formulas:

$$\begin{cases} H_{tr}^0(\mathcal{C}) \cong \mathbb{Z}, \\ H_{tr}^1(\mathcal{C}) \cong \mathbb{Z} + \mathbb{Z}\theta_1 + \dots + \mathbb{Z}\theta_{2g-1}, \\ H_{tr}^2(\mathcal{C}) \cong \mathbb{Z}, \end{cases} \quad (32)$$

where $\theta_i \in \mathbb{R}$ are algebraically independent integers of a number field of degree $2g$.

Proof. It is known that the Serre C^* -algebra of the (generic) complex algebraic curve \mathcal{C} is isomorphic to a *toric AF*-algebra \mathbb{A}_θ , see [11] for the notation and details. Moreover, up to a scaling constant $\mu > 0$, it holds

$$\tau_*(K_0(\mathbb{A}_\theta \otimes \mathcal{K})) = \begin{cases} \mathbb{Z} + \mathbb{Z}\theta_1 & \text{if } g = 1 \\ \mathbb{Z} + \mathbb{Z}\theta_1 + \dots + \mathbb{Z}\theta_{6g-7} & \text{if } g > 1, \end{cases} \quad (33)$$

where constants $\theta_i \in \mathbb{R}$ parametrize the moduli (Teichmüller) space of curve \mathcal{C} , *ibid.* If \mathcal{C} is defined over a number field k , then each θ_i is algebraic and their total number is equal to $2g - 1$. (Indeed, since $Gal(\bar{k} | k)$ acts on the torsion points of $\mathcal{C}(k)$, it is easy to see that the endomorphism ring of $\mathcal{C}(k)$ is non-trivial. Because such a ring is isomorphic to the endomorphism ring of jacobian $Jac \mathcal{C}$ and $\dim_{\mathbb{C}} Jac \mathcal{C} = g$, one concludes that $End \mathcal{C}(k)$ is a \mathbb{Z} -module of rank $2g$ and each θ_i is an algebraic number.) After scaling by a constant $\mu > 0$, one gets

$$H_{tr}^1(\mathcal{C}) := \tau_*(K_0(\mathbb{A}_\theta \otimes \mathcal{K})) = \mathbb{Z} + \mathbb{Z}\theta_1 + \dots + \mathbb{Z}\theta_{2g-1} \quad (34)$$

Because $H^0(\mathcal{C}) \cong H^2(\mathcal{C}) \cong \mathbb{Z}$, one obtains the rest of formulas (32). \square

Remark 11 If $k \cong \mathbb{Q}$, then $\Gamma \cong \Gamma(N)$ is the principal congruence subgroup of level N , since $\mathcal{C}(\mathbb{Q}) \cong X_{\Gamma(N)}$ for some integer N . As explained, the Petersson inner product on $S_2(\Gamma(N))$ gives rise to a positive-definite Hermitian

form φ on the cohomology group $H^1(\mathcal{C}) \cong \mathbb{Z}^{2g}$. Note that the form φ can be obtained from the classical Riemann's bilinear relations for the periods of curve \mathcal{C} ; this yields Weil's proof of the Riemann hypothesis for function $Z_{\mathcal{C}}(t)$.

Remark 12 Notice the cardinality of the set $\mathcal{C}(\mathbb{F}_q)$ is given to the formula

$$|\mathcal{C}(\mathbb{F}_q)| = 1 + q - \text{tr}(\omega) = 1 + q - \sum_{i=1}^{2g} \lambda_i, \quad (35)$$

where λ_i are the eigenvalues of the Frobenius endomorphism $\omega \in \text{End}(H_{tr}^1(\mathcal{C}))$.

Example 2 The case $g = 1$ is particularly instructive; for the sake of clarity, we shall consider elliptic curves having complex multiplication. Let $\mathcal{E}(\mathbb{F}_q)$ be the reduction modulo q of an elliptic with complex multiplication by the ring of integers of an imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$, see e.g. [Silverman 1994] [16], Chapter 2. It is known, that in this case the trace cohomology formulas (32) take the form

$$\begin{cases} H_{tr}^0(\mathcal{E}(\mathbb{F}_q)) \cong \mathbb{Z}, \\ H_{tr}^1(\mathcal{E}(\mathbb{F}_q)) \cong \mathbb{Z} + \mathbb{Z}\sqrt{d}, \\ H_{tr}^2(\mathcal{E}(\mathbb{F}_q)) \cong \mathbb{Z}. \end{cases} \quad (36)$$

We shall denote by $\psi(\mathfrak{P}) \in \mathbb{Q}(\sqrt{-d})$ the Grössencharacter of the prime ideal \mathfrak{P} over p , see [Silverman 1994] [16], p. 174. It is easy to see, that in this case the Frobenius endomorphism $\omega \in \text{End}(H_{tr}^1(\mathcal{E}(\mathbb{F}_q)))$ is given by the formula

$$\omega = \frac{1}{2} [\psi(\mathfrak{P}) + \overline{\psi(\mathfrak{P})}] + \frac{1}{2} \sqrt{(\psi(\mathfrak{P}) + \overline{\psi(\mathfrak{P})})^2 + 4q} \quad (37)$$

and the corresponding eigenvalues

$$\begin{cases} \lambda_1 = \omega = \frac{1}{2} [\psi(\mathfrak{P}) + \overline{\psi(\mathfrak{P})}] + \frac{1}{2} \sqrt{(\psi(\mathfrak{P}) + \overline{\psi(\mathfrak{P})})^2 + 4q}, \\ \lambda_2 = \bar{\omega} = \frac{1}{2} [\psi(\mathfrak{P}) + \overline{\psi(\mathfrak{P})}] - \frac{1}{2} \sqrt{(\psi(\mathfrak{P}) + \overline{\psi(\mathfrak{P})})^2 + 4q}. \end{cases} \quad (38)$$

Using formula (35), one gets the following equation

$$|\mathcal{E}(\mathbb{F}_q)| = 1 - (\lambda_1 + \lambda_2) + q = 1 - \psi(\mathfrak{P}) - \overline{\psi(\mathfrak{P})} + q, \quad (39)$$

which coincides with the well-known expression for $|\mathcal{E}(\mathbb{F}_q)|$ in terms of the Grössencharacter, see e.g. [Silverman 1994] [16], p. 175.

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